LEARNING OBJECTIVES

In this section you will:

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

2.7 LINEAR INEQUALITIES AND ABSOLUTE VALUE INEQUALITIES

Figure 1

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in **Figure 2**. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

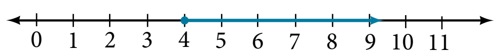


Figure 2

We can use set-builder notation: $\{x|x \geq 4\}$, which translates to "all real numbers x such that x is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $(-\infty, 1]$, all real numbers less than and including 1. **Table 1** outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b, but not including a or b	$\{x\|\ a < x < b\}$	(a, b)
All real numbers greater than a, but not including a	$\{x\|\ x > a\}$	(a, ∞)
All real numbers less than b, but not including b	$\{x\|\ x < b\}$	$(-\infty, b)$
All real numbers greater than a, including a	$\{x\|\ x \geq a\}$	$[a, \infty)$

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers less than b, including b	$\{x \mid x \leq b\}$	$(-\infty, b]$
All real numbers between a and b, including a	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between a and b, including b	$\{x \mid a < x \leq b\}$	$(a, b]$
All real numbers between a and b, including a and b	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x \mid x < a \text{ and } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x \mid x \text{ is all real numbers}\}$	$(-\infty, \infty)$

Table 1

Example 1 **Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a***

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Try It #1

Use interval notation to indicate all real numbers between and including -3 and 5.

Example 2 **Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b***

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1, which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

Try It #2

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

properties of inequalities

Addition Property	If $a < b$, then $a + c < b + c$.
Multiplication Property	If $a < b$ and $c > 0$, then $ac < bc$.
	If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

Example 3 **Demonstrating the Addition Property**

Illustrate the addition property for inequalities by solving each of the following:

a. $x - 15 < 4$ **b.** $6 \geq x - 1$ **c.** $x + 7 > 9$

Solution The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

$$x - 15 < 4$$
$$x - 15 + 15 < 4 + 15 \quad \text{Add 15 to both sides.}$$
$$x < 19$$

b.

$$6 \geq x - 1$$
$$6 + 1 \geq x - 1 + 1 \quad \text{Add 1 to both sides.}$$
$$7 \geq x$$

c.

$$x + 7 > 9$$
$$x + 7 - 7 > 9 - 7 \quad \text{Subtract 7 from both sides.}$$
$$x > 2$$

Try It #3

Solve: $3x - 2 < 1$.

Example 4 Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. $3x < 6$ **b.** $-2x - 1 \geq 5$ **c.** $5 - x > 10$

Solution

a.

$$3x < 6$$
$$\frac{1}{3}(3x) < (6)\frac{1}{3}$$
$$x < 2$$

b.

$$-2x - 1 \geq 5$$
$$-2x \geq 6$$
$$\left(-\frac{1}{2}\right)(-2x) \geq (6)\left(-\frac{1}{2}\right) \quad \text{Multiply by } -\frac{1}{2}.$$
$$x \leq -3 \quad \text{Reverse the inequality.}$$

c.

$$5 - x > 10$$
$$-x > 5$$
$$(-1)(-x) > (5)(-1) \quad \text{Multiply by } -1.$$
$$x < -5 \quad \text{Reverse the inequality.}$$

Try It #4

Solve: $4x + 7 \geq 2x - 3$.

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example 5 Solving an Inequality Algebraically

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution Solving this inequality is similar to solving an equation up until the last step.

$$13 - 7x \geq 10x - 4$$
$$13 - 17x \geq -4 \quad \text{Move variable terms to one side of the inequality.}$$
$$-17x \geq -17 \quad \text{Isolate the variable term.}$$
$$x \leq 1 \quad \text{Dividing both sides by } -17 \text{ reverses the inequality.}$$

The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1.

Try It #5

Solve the inequality and write the answer using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

Example 6 Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

Solution We begin solving in the same way we do when solving an equation.

$$-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x.$$
$$-\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8} \quad \text{Put variable terms on one side.}$$
$$-\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8} \quad \text{Write fractions with common denominator.}$$
$$-\frac{17}{12}x \geq -\frac{5}{8}$$
$$x \leq -\frac{5}{8}\left(-\frac{12}{17}\right) \quad \text{Multiplying by a negative number reverses the inequality.}$$
$$x \leq \frac{15}{34}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

Try It #6

Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Example 7 Solving a Compound Inequality

Solve the compound inequality: $3 \leq 2x + 2 < 6$.

Solution The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$3 \leq 2x + 2 \quad \text{and} \quad 2x + 2 < 6$$
$$1 \leq 2x \qquad 2x < 4$$
$$\frac{1}{2} \leq x \qquad x < 2$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \le x < 2$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$3 \le 2x + 2 < 6$$

$1 \le 2x < 4$ Isolate the variable term, and subtract 2 from all three parts.

$\frac{1}{2} \le x < 2$ Divide through all three parts by 2.

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

Try It #7

Solve the compound inequality: $4 < 2x - 8 \le 10$.

Example 8 Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Solution Let's try the first method. Write two inequalities:

$3 + x > 7x - 2$	and	$7x - 2 > 5x - 10$
$3 > 6x - 2$		$2x - 2 > -10$
$5 > 6x$		$2x > -8$
$\frac{5}{6} > x$		$x > -4$
$x < \frac{5}{6}$		$-4 < x$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See **Figure 3**.

Figure 3

Try It #8

Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x, as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, |A| \le B, |A| > B, \text{or } |A| \ge B,$$

Where A, and sometimes B, represents an algebraic expression dependent on a variable x. Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of x-values such that the distance between x and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x - 600|$, and therefore, $|x - 600| \leq 200$ or

$$-200 \leq x - 600 \leq 200$$
$$-200 + 600 \leq x - 600 + 600 \leq 200 + 600$$
$$400 \leq x \leq 800$$

This means our returns would be between \$400 and \$800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

absolute value inequalities

For an algebraic expression X, and $k > 0$, an **absolute value inequality** is an inequality of the form

$$|X| < k \text{ is equivalent to } -k < X < k$$
$$|X| > k \text{ is equivalent to } X < -k \text{ or } X > k$$

These statements also apply to $|X| \leq k$ and $|X| \geq k$.

Example 9 Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in **Figure 4**, to represent the condition to be satisfied.

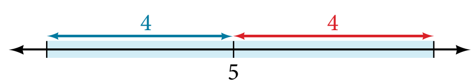


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x - 5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x - 5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$x - 5 \leq 4 \quad \text{and} \quad x - 5 \geq -4$$
$$x \leq 9 \qquad\qquad x \geq 1$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x - 5| \leq 4$ is equivalent to $[1, 9]$ in interval notation.

Try It #9

Describe all x-values within a distance of 3 from the number 2.

Example 10 Solving an Absolute Value Inequality

Solve $|x - 1| \leq 3$.

Solution

$$|x - 1| \leq 3$$
$$-3 \leq x - 1 \leq 3$$
$$-2 \leq x \leq 4$$
$$[-2, 4]$$

Example 11 Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\frac{1}{2}|4x - 5| + 3$, determine the x-values for which the y-values are negative.

Solution We are trying to determine where $y < 0$, which is when $-\frac{1}{2}|4x - 5| + 3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x - 5| < -3 \qquad \text{Multiply both sides by –2, and reverse the inequality.}$$
$$|4x - 5| > 6$$

Next, we solve for the equality $|4x - 5| = 6$.

$$\begin{aligned} 4x - 5 &= 6 & \text{or} \qquad 4x - 5 &= -6 \\ 4x &= 11 & 4x &= -1 \\ x &= \frac{11}{4} & x &= -\frac{1}{4} \end{aligned}$$

Now, we can examine the graph to observe where the y-values are negative. We observe where the branches are below the x-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$, and that the graph opens downward. See **Figure 5**.

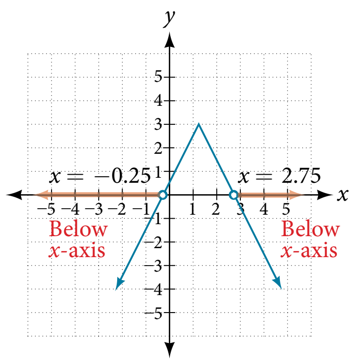


Figure 5

Try It #10

Solve $-2|k - 4| \leq -6$.

Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

- Interval Notation (http://openstaxcollege.org/l/intervalnotn)
- How to Solve Linear Inequalities (http://openstaxcollege.org/l/solvelinineq)
- How to Solve an Inequality (http://openstaxcollege.org/l/solveineq)
- Absolute Value Equations (http://openstaxcollege.org/l/absvaleq)
- Compound Inequalities (http://openstaxcollege.org/l/compndineqs)
- Absolute Value Inequalities (http://openstaxcollege.org/l/absvalineqs)

2.7 SECTION EXERCISES

VERBAL

1. When solving an inequality, explain what happened from Step 1 to Step 2:
Step 1 $-2x > 6$
Step 2 $x < -3$

2. When solving an inequality, we arrive at:
$x + 2 < x + 3$
$2 < 3$
Explain what our solution set is.

3. When writing our solution in interval notation, how do we represent all the real numbers?

4. When solving an inequality, we arrive at:
$x + 2 > x + 3$
$2 > 3$
Explain what our solution set is.

5. Describe how to graph $y = |x - 3|$

ALGEBRAIC

For the following exercises, solve the inequality. Write your final answer in interval notation

6. $4x - 7 \leq 9$

7. $3x + 2 \geq 7x - 1$

8. $-2x + 3 > x - 5$

9. $4(x + 3) \geq 2x - 1$

10. $-\frac{1}{2}x \leq -\frac{5}{4} + \frac{2}{5}x$

11. $-5(x - 1) + 3 > 3x - 4 - 4x$

12. $-3(2x + 1) > -2(x + 4)$

13. $\frac{x + 3}{8} - \frac{x + 5}{5} \geq \frac{3}{10}$

14. $\frac{x - 1}{3} + \frac{x + 2}{5} \leq \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

15. $|x + 9| \geq -6$

16. $|2x + 3| < 7$

17. $|3x - 1| > 11$

18. $|2x + 1| + 1 \leq 6$

19. $|x - 2| + 4 \geq 10$

20. $|-2x + 7| \leq 13$

21. $|x - 7| < -4$

22. $|x - 20| > -1$

23. $\left|\frac{x - 3}{4}\right| < 2$

For the following exercises, describe all the x-values within or including a distance of the given values.

24. Distance of 5 units from the number 7

25. Distance of 3 units from the number 9

26. Distance of 10 units from the number 4

27. Distance of 11 units from the number 1

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

28. $-4 < 3x + 2 \leq 18$

29. $3x + 1 > 2x - 5 > x - 7$

30. $3y < 5 - 2y < 7 + y$

31. $2x - 5 < -11$ or $5x + 1 \geq 6$

32. $x + 7 < x + 2$

GRAPHICAL

For the following exercises, graph the function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

33. $|x-1|>2$ **34.** $|x+3|\geq 5$ **35.** $|x+7|\leq 4$ **36.** $|x-2|<7$ **37.** $|x-2|<0$

For the following exercises, graph both straight lines (left-hand side being y_1 and right-hand side being y_2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the y-values of the lines.

38. $x+3<3x-4$ **39.** $x-2>2x+1$ **40.** $x+1>x+4$ **41.** $\frac{1}{2}x+1>\frac{1}{2}x-5$

42. $4x+1<\frac{1}{2}x+3$

NUMERIC

For the following exercises, write the set in interval notation.

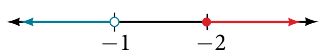
43. $\{x|-1<x<3\}$ **44.** $\{x|x\geq 7\}$ **45.** $\{x|x<4\}$ **46.** $\{x|\ x$ is all real numbers$\}$

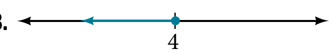
For the following exercises, write the interval in set-builder notation.

47. $(-\infty, 6)$ **48.** $(4, \infty)$ **49.** $[-3, 5)$ **50.** $[-4, 1]\cup[9, \infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

51. 

52. 

53. 

TECHNOLOGY

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter **Y2=** the right-hand side. Entering the absolute value of an expression is found in the **MATH** menu, **Num**, **1:abs(**. Find the points of intersection, recall (**2nd CALC 5:intersection**, **1st curve, enter, 2nd curve, enter, guess, enter**). Copy a sketch of the graph and shade the x-axis for your solution set to the inequality. Write final answers in interval notation.

54. $|x+2|-5<2$ **55.** $-\frac{1}{2}|x+2|<4$ **56.** $|4x+1|-3>2$ **57.** $|x-4|<3$

58. $|x+2|\geq 5$

EXTENSIONS

59. Solve $|3x+1|=|2x+3|$

60. Solve $x^2-x>12$

61. $\frac{x-5}{x+7}\leq 0, x\neq -7$

62. $p=-x^2+130x-3{,}000$ is a profit formula for a small business. Find the set of x-values that will keep this profit positive.

REAL-WORLD APPLICATIONS

63. In chemistry the volume for a certain gas is given by $V=20T$, where V is measured in cc and T is temperature in °C. If the temperature varies between 80°C and 120°C, find the set of volume values.

64. A basic cellular package costs \$20/mo. for 60 min of calling, with an additional charge of \$0.30/min beyond that time. The cost formula would be $C=\$20+.30(x-60)$. If you have to keep your bill lower than \$50, what is the maximum calling minutes you can use?